Hasegawa-Wakatani Equation Deduction

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1 Introduction

The Hasegawa-Wakatani equation is used to describe the plasma behavior in a strong and uniform magnetic field. This scenario is usually observed in the Tokamak nuclear fusion device. Due to the strong magnetic field, most of the charged particles are confined and do a periodic motion on a circular orbit. However, due to the collision between the particles, turbulence could be generated and magnified to a scale of $\sim 1\%$ and cause some particle to escape. To decrease the amount of the escaped particles which could damage the Tokamak device, it is better to reduce the turbulence.

2 A Particle in the Electric-Magnetic Field

2.1 Gyroradius

If a charged particle is moving in the magnetic field, then the magnetic force on this particle is:

$$\vec{F} = q\vec{v} \times \vec{B} \ , \tag{1}$$

where $\vec{v}$ is the particle velocity, $\vec{B}$ is the magnetic field strength, q is the particle charge. The direction of this force is perpendicular to the velocity direction and the magnetic field direction, therefore a cross-product ($\times$) is used in this equation. For example, if the particle is positively charged, and moving towards positive y direction, and the magnetic field is towards positive z direction, then the force is towards x direction (which, in my high-school textbook, is described as left-hand rule).
Because the force is always perpendicular to the velocity, the charged particle in the magnetic field undergo circular motion, the radius of the orbit is (gyroradius):

$$r = \frac{m\vec{v} \times \vec{B}}{|B|^2} \ , \tag{2}$$

where m is the mass of the particle. The gyrofrequency is:

$$\omega = \frac{qB}{m} \ . \tag{3}$$

Based on Equation 2, we could find a few simple rules in the Tokamak device: (1) Particles could easily move parallel to the magnetic field, especially electron which has a low mass; (2) The mass of ions are usually several thousands times of the mass of electrons, therefore ions have larger gyroradii than electrons, and less confined than electrons.

2.2 Electrostatic Field

If a charged particle is in a electrostatic field, then the electrostatic force on this particle is:

$$\vec{F} = q\vec{E} = -q\nabla\phi \ , \tag{4}$$

where $\vec{E}$ is the electrostatic field strength, and is equal to the spatial gradient of the electrostatic field potential ϕ. The direction of the force is parallel to the direction of $\vec{E}$. When a particle is exposed to a electrostatic field and a magnetic field, then there exists a velocity for the particle that balance the forces and keep the particle to move straight:

$$\vec{v}_E = -\frac{\nabla\phi \times \vec{B}}{B^2} \ . \tag{5}$$

This velocity is defined as drift velocity.

2.3 Polarization Drift

When the electrostatic field id changing with time, then the particle has a polarization drift velocity:

$$\vec{v}_p = \pm\frac{1}{\omega B}\frac{\partial \vec{E}}{\partial t} = \pm\frac{1}{\omega B}\frac{\partial \nabla\phi}{\partial t} \ , \tag{6}$$

where $\omega = qB/m$ is the gyrofrequency, the symbol of this equation follows the charge of the particle. The deduction of this equation could be found in [2], Eq 2-60 to Eq 2-66 (I am reading it in a Chinese version, hopefully they are the same).

3 Plasma Fluid Dynamics

3.1 The Convective Derivative

When treating plasma as fluid, we assume the plasma has a velocity field, which means at each spatial point of the plasma, there is a single velocity to describe the fluid element, and all the particles inside the same fluid element should share a same velocity. Therefore, the convective derivative, as an operator, is defined:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\nabla) = \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} \tag{7}$$

Using this convective derivative, we could focus on a certain fluid element and discuss how the physical quantities are changing in this fluid element.
Recall that this convective derivative is also used in the radiative transfer equation [1]:

$$\frac{1}{c}\frac{\partial u}{\partial t} + \omega\cdot\nabla_x u + ku + \sigma\,(some\ scattering\ terms) = f \tag{8}$$

We can see the first and the second term in the equation is the convective derivative, it tells the change this certain bunch of photon is due to the absorption term, the scattering term, and the emission term it follows. Therefore, the velocity of the ions in the Tokamak device could be described as:

$$\frac{d\vec{u}}{dt} = -\frac{e}{m}\nabla\phi + \frac{e}{m}\vec{u}\times\vec{B} - \nabla p_i - \nabla\Pi + F \ , \tag{9}$$

where the left hand side is the convective derivative of the velocity field. In the right hand side, the first term is the acceleration due to the electrostatic force, the second term is the acceleration due to the magnetic force, the third term is the pressure due to the ion-ion collision, the fourth term is viscosity, the fifth term is external force. A similar equation could also be written for electrons.

4 Hasegawa-Wakatani Equations

4.1 Assumptions and Approximations

A system of two equations of 9 to describe the activity of ions and electrons is apparently too complicated, a few assumptions are introduced.

- The number of ions and the number of electrons are conserved, so no need to calculate the ionization and recombination.
- The ion mass is much larger than electron mass and some forces are ignored. In reality, a proton (H^+ ion) is 1836 times more massive than an electron.
- The plasma is "warm", which means a high electron temperature and a low ion temperature, and the electron velocity is much faster than the ion velocity.
- The uniform background magnetic field is extremely strong, and the magnetic field induced by the motion of charged particles (some Maxwell equation deduction not mentioned in this note, please find a hard-core electrodynamic textbook) is negligible.

4.2 Ion Velocity

We firstly focus on the velocity equation of ion fluid, based on the as-mentioned assumptions, in the Equation 9, the external force $F = 0$; the pressure term is also zero; the viscosity term is:

$$\nabla \Pi = -\mu \nabla^2 \vec{u} \ , \tag{10}$$

where μ is the viscosity coefficient. Therefore, Equation 9 is re-written as:

$$\frac{d\vec{u}}{dt} = -\frac{e}{m}\nabla\phi + \frac{e}{m}\vec{u} \times \vec{B} + \mu\nabla^2\vec{u} \tag{11}$$

The zeroth order solution of the ion velocity is:

$$\vec{u}_0 = -\nabla\phi \times \frac{\vec{B}}{B^2} \ , \tag{12}$$

which is the ion drift velocity discussed in Equation 5. The first order solution of the ion velocity is:

$$\vec{u}_1 = -\frac{1}{\omega B}\frac{\partial \nabla\phi}{\partial t} - \frac{\mu}{\omega B}\nabla^2(\nabla\phi), \tag{13}$$

which is due to the polarized drift velocity discussed in Equation 6 and the viscosity term, ω is the gyrofrequency. I am not sure, but it seems plugging $\vec{u} = \vec{u}_0 + \vec{u}_1$ into Equation 11 could produce a 0=0 to prove the correctness of the solution.

4.3 Charge Conservation

When a non-zero electrostatic potential ϕ exist in the plasma due to random fluctuations or external force, the ions are drifting from $\phi > 0$ area to $\phi < 0$ area, and the electrons are drifting from $\phi < 0$ area to $\phi > 0$ area, so that the electrostatic potential is flattened.
Because the zeroth-order solution (Equation 12) for both electrons and ions are always perpendicular to the potential gradient $\nabla\phi$ (obviously, due to the cross product), this term is not contributing to this process.
Because the mass of electron is very small comparing to the mass of ion, the first-order solution (Equation 13) for electrons is not significant and can be ignored.
Therefore, the ions are drifting with $\vec{u}_1$ to balance the charge, and the current is:

$$\vec{J}_\perp = n|e|\vec{u}_1 \ . \tag{14}$$

Note that this current direction is only perpendicular to the magnetic field. Although the electrons are not drifting in the perpendicular direction of the magnetic field, they are allowed to move parallel to the magnetic field. Because the electron mass is much less then the ion mass and the electron temperature is much higher than the ion temperature, the electrons are allowed to move much faster parallel to the magnetic field to flatten the potential. Therefore, the parallel current due to the ion drift is not significant comparing to electron. Moreover, we assume the ion density is not changing much in this question because it is just small turbulence, so the major contribution of the current is the velocity of the ions instead of the actual change of the density, and it should be safe to use a non-turbulence density n here.
Based on the charge conservation equation, the divergence of current is zero:

$$\nabla \cdot \vec{J} = 0 \ . \tag{15}$$

Therefore, the non-zero perpendicular current divergence due to the drift of ions should be balanced by the parallel current due to the electrons:

$$\nabla_\perp \cdot \vec{J}_\perp = -\nabla_\parallel J_\parallel . \tag{16}$$

4.4 The Velocity of Electrons

We assume the electron collision forms a isothermal electron fluid, which means the temperature T_e is not changing. Therefore, the pressure of electron is propotional to the electron number density:

$$p_e = n_e k_B T_e \ , \tag{17}$$

where k_B is the Boltzmann constant. Therefore, we could write down the following equation of the parallel current density due to the motion of electrons:

$$J_\parallel = \frac{k_B T_e}{e\eta} \nabla_\parallel \left(\frac{n_e - n}{n} - \frac{e\phi}{T_e k_B} \right) \ . \tag{18}$$

Note that n is the number density of electron or ion if there is no turbulence, and even with turbulence as we calculated, n_e is still quite close to n, and η is the resistivity. The first term $(n_e - n)/n$ is the contribution due to the pressure gradient, the second term is the contribution due to the electrostatic field. The resisitivity η represents the collision rate between the moving electrons and stationary ions, and is a constant in this question.

The velocity of electron could be deduced from the electron current density $J_\parallel$:

$$u_{e,\parallel} = -\frac{J_\parallel}{n} \tag{19}$$

again, electrons only have parallel velocity, and they are orbiting very small circles perpendicularly.

4.5 The HW Equation

Now, we focus on the equation of the current divergence Equation 16, firstly use Equation 13 and Equation 14 to replace the $J_\perp$ term in the left hand side, secondly use Equation 18 in the right hand side, the first HW equation is:

$$-\frac{ne}{\omega B} \nabla_\perp \cdot \left(\frac{\partial \nabla\phi}{\partial t} + \mu \nabla^2 (\nabla\phi) \right) = -\frac{k_B T_e}{e\eta} \nabla_\parallel^2 \left(\frac{n_e - n}{n} - \frac{e\phi}{T_e k_B} \right) \ . \tag{20}$$

The second HW equation is based on the conservation of the number of electrons:

$$\frac{\partial n_e}{\partial t} + n \nabla_\parallel u_{e,\parallel} = 0 \ , \tag{21}$$

then we input the electron velocity into this equation and get:

$$\frac{\partial (n_e - n)}{\partial t} = \frac{k_B T_e}{e\eta} \nabla_\parallel \left(\frac{n_e - n}{n} - \frac{e\phi}{T_e k_B} \right) \ . \tag{22}$$

So, this is the physics illustration of the Hasegawa-Wakatani Equation. One last thing, in the numerical simulations, the viscosity coefficient μ is usually very small to make the solution more stable, and μ could be zero in reality.

There are still some variations (i.e. assuming the background density is a linear profile, the usage of poission bracket) of the equation not discussed. The physical quantities e, k_B are still perserved in the equations, others may use some simple coefficients.

References

[1] Siddhartha Mishra and Roberto Molinaro. Physics informed neural networks for simulating radiative transfer. jqsrt, 270:107705, August 2021.

[2] G. Rowlands. Introduction to Plasma Physics and Controlled Fusion, Volume 1 (2nd Edition), F. F. Chen. Plenum, 1984, £23.30, xv + 421 pages. Journal of Plasma Physics, 44(3):547, January 1990.